Discrete Applied Mathematics 6 (1983) 237-241
North-Holland

RECOGNIZING SIGN SOLVABLE GRAPHS*

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Received 1 June 1981
Revised 12 April 1982

A characterization of sign solvable graphs is presented together with a polynomial algorithm for recognizing such graphs.

1. Introduction

The analysis of qualitative linear systems of equations, in which the signs $-$, 0 or $+$ of all coefficients are known, but not their magnitudes, was initiated in 1947 by Samuelson, in his classical treatise *Foundations of Economic Analysis*. Economists, mathematicians, chemists, and ecologists have investigated the subject, concentrating on issues of *stability* and of *sign solvability* – see Quirk [9] and Maybee [8] for recent surveys. Let $Ay=b$ denote a square qualitative system; it is said to be *sign solvable* if and only if the existence of a solution y and the signs of all variables in that solution are entirely determined by the signs of the coefficients of A and b. The sign solvability problem consists in determining necessary and sufficient conditions for this to hold; in constructive form it amounts to asking for an algorithm for recognizing whether any given qualitative linear system is sign solvable or not. Lancaster [4] stated that problem in 1962 and a non-constructive solution was provided by Bassett, Maybee and Quirk [1] in 1968. Maybee [7] recently introduced the class of *sign solvable graphs* with an aim at obtaining a constructive solution, through a reformulation of the problem in terms of signed graphs. The purpose of the present paper is to continue Maybee's research by providing a characterization of sign solvable graphs, which leads to a polynomial algorithm for recognizing such graphs. The latter's time complexity is $O(nm)$ and space complexity $O(n+m)$ where n and m denote the numbers of vertices and arcs respectively of the graphs under study.

Following Klee and Ladner [3] a square qualitative system $Ay=b$ is said to be *strongly sign solvable* if and only if it is sign solvable and no component of y takes the value 0. In a very recent paper, Manber [5] has translated in terms of signed graphs both the sign solvability and the strong sign solvability problem.

*This paper was written during the author's visit to the Graduate School of Business, University of Pittsburgh. Support of the GSB is gratefully acknowledged.

0166-218X/83/$3.00

2. A characterization of sign solvable graphs

Let us first recall some definitions, following the terminology of Berge [2]. *Arc, path* and *circuit* are concepts implying an orientation; a path or circuit is *elementary* if and only if it does not contain twice the same vertex. A *partial graph* $G'=(X, U')$ of a graph $G=(X, U)$ is a graph with the same vertex set X as G and a subset $U' \subset U$ of its arcs; a *subgraph* $G_B=(B, U_B)$ of $G=(X, U)$ is a graph having a subset $B \subset X$ of the vertices of G and as arcs those of G both end vertices of which belong to B. A *signed graph* G is a graph to each arc of which either a sign + or a sign − has been given. Such a graph is naturally associated to a square qualitative matrix A for interpreting the latter as a *signed adjacency matrix:* $a_{ij}=+$ iff a positive arc goes from x_i to x_j, $a_{ij}=0$ iff no arc goes from x_i to x_j and $a_{ij}=-$ iff a negative arc goes from x_i to x_j. Of particular interest will be the *positive partial graph* $G^+=(X, U^+)$ of G, obtained by retaining only all positive arcs of G. The *sign of a path* or *circuit* is the product of the signs of its arcs. Let us call, following Maybee [8], *distinguished vertex* any vertex of G which is the terminal vertex of elementary paths with a positive sign only. We then have

Theorem 2.1 (Maybee). *The signed graph G is sign solvable if and only if*
(i) *G contains no elementary positive circuit.*
(ii) *G contains a set S of distiguished vertices.*
(iii) *Each strongly connected component of G contains a vertex of S.*

Conditions (i) and (ii) are difficult to check. Indeed the number of elementary circuits of G is not polynomially bounded and, moreover, G may contain positive circuits but no elementary positive circuits. Similarly, a vertex x_i may be the terminal vertex of a negative path but of no elementary negative path. We therefore propose the following characterization of sign solvable graphs.

Theorem 2.2. *The signed graph G is sign solvable if and only if*
(a) *The positive partial graph G does contain no circuit.*
(b) *Each strongly connected component of G contains a distinguished vertex.*

Proof. Clearly, condition (b) is equivalent to conditions (ii) and (iii). Condition (i) implies condition (a) as any circuit in G^+ is positive and the existence of such a circuit would imply that of an elementary positive circuit. It remains therefore to show that conditions (a) and (b) imply condition (i). Assume by contradiction (a) and (b) hold but G contains an elementary positive circuit C. From (a), C must contain at least two negative arcs. Consider the strongly connected component of G containing C; from (b) it contains a distinguished vertex, say x_k. Let P denote a shortest path going from a vertex of C to x_k and x_j the vertex of C on that path (note that x_k and x_j may coincide). P being a shortest path is elementary and cannot contain a negative arc otherwise the subpath of P beginning at the initial vertex of the negative arc of P closest to x_k would be negative, contradicting the assumption that x_k is a distinguished vertex. Consider now the shortest path P' in C going from the initial vertex of a negative arc to x_j. The path consisting of P' followed by P is elementary for if P' and P had a common vertex different from x_j, say x_t, it would be closer to x_k than x_j, a contradiction. But that path is negative, which contradicts the assumption that x_k is a distinguished vertex. □

3. An algorithm for sign solvability

The characterization of sign solvable graphs given in the previous section suggests the following algorithm to recognize such graphs:

Step 1: Consider the positive partial graph G^+ of G. Check if it is circuit free. If not, stop; G is not sign solvable.

Step 2: Determine the strongly connected components of G.

Step 3: Determine all vertices of G which are not distinguished. To this effect consider in turn each vertex $x_j \in X$ which is the initial vertex of at least one negative arc; construct the subgraph $G^+_{X-\{x_j\}} = (X - \{x_j\},\ U^+_{X-\{x_j\}})$ of G^+ obtained by deleting x_j and all adjacent arcs; label the vertices of $G^+_{X-\{x_j\}}$ which are terminal vertices of negative arcs with x_j as initial vertex in G; iteratively label all vertices which are terminal vertices of arcs with labelled initial vertices in $G^+_{X-\{x_j\}}$.

Step 4: All vertices unlabelled in Step 3 are distinguished. Check that each strongly connected component of G contains one such vertex; if so, G is sign solvable and otherwise not.

Theorem 3.1. *The above algorithm recognizes sign solvable graphs and has a worst-case time complexity* $O(mn)$ *and space complexity* $O(n+m)$, *where* $n = |X|$ *and* $m = |U|$.

Proof. Consider first the algorithm's correctness. Step 1 corresponds to condition (a) of Theorem 2.2. Steps 2 to 3 correspond to condition (b) in view of the observation that any negative elementary path of G contains a negative elementary subpath with a single negative arc which is its first one. All such paths consist therefore of a negative arc from a vertex x_j followed by a path in G^+ *not going through* x_j. In Step 3, these paths are obtained by labelling in the subgraphs $G^+_{X-\{x_j\}}$. Regarding complexity, we choose to represent G by the linked lists of positive and of negative arcs with x_j as initial vertex for $j = 1, 2, \ldots, n$. Then Step 1 can be performed in $O(m)$ time with Marimont's [8] algorithm. Strongly connected components can also be found in $O(m)$ time with the depth-first search method of Tarjan [13]. Labelling the terminal vertices from negative arcs in Step 3 takes $O(m)$ time for all vertices and labelling in $G^+_{X-\{x_j\}}$ $O(m)$ time for each vertex x_j, i.e., $O(mn)$ time in all. Step

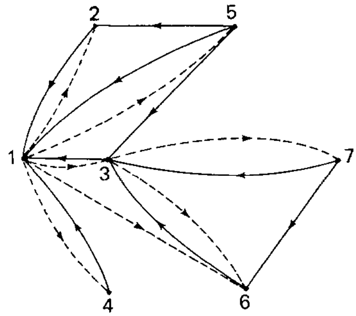

Fig. 1. G.

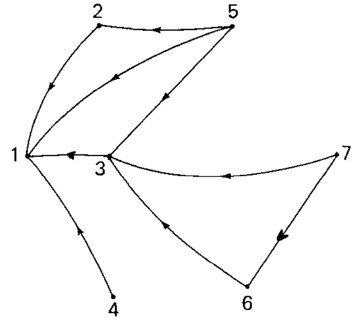

Fig. 2. G^+

4 takes $O(n)$ operations. As the graph G is noted by lists of length $O(n+m)$ and labelling in Steps 1 to 3 takes $O(n)$ space, the space complexity is $O(n+m)$. □

The dominant step for time is clearly the third one. In order to accelerate the algorithm lists of the vertices of each strong component of G could be kept after Step 1 and vertices eliminated from those lists as soon as they are labelled; if one list becomes empty the algorithm may be stopped.

4. An example

To illustrate the algorithm of the previous section we consider an example from Maybee [7]. The signed graph is reproduced in Fig. 1, with positive arcs in full and negative arcs in broken lines. The positive partial graph G^+ is shown in Fig. 2. Marimont's algorithm provides the following ranking of vertices such that each arc goes from a vertex with lower rank to one with higher rank: $x_4, x_5, x_7, x_2, x_6, x_3, x_1$; hence G^+ has no circuit. G has a single connected component.

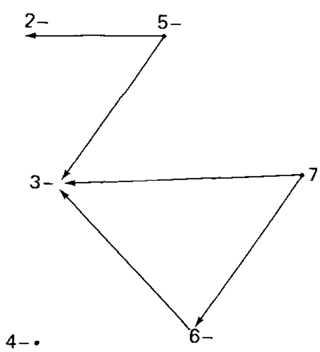

Fig. 3. $G^+_{X-\{x_1\}}$.

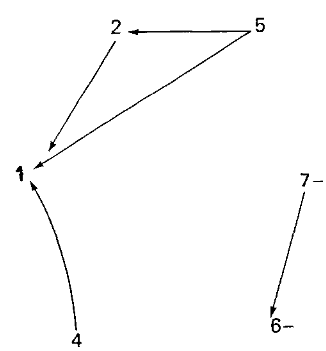

Fig. 4. $G^+_{X-\{x_3\}}$.

Only x_1 and x_3 are initial vertices of negative arcs of G; the subgraphs $G^+_{X-\{x_1\}}$ and $G^+_{X-\{x_3\}}$ are represented in Figs. 3 and 4 with labelled vertices marked $-$.

It is seen that x_1 is the only distinguished vertex and that G is sign solvable.

Acknowledgements

The author thanks R. Manber and an anonymous referee for their helpful comments.

References

[1] L. Bassett, J. Maybee and J. Quirk, Qualitative economics and the scope of the correspondence principle, Econometrica 26 (1968) 544–563.

[2] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1973).

[3] V. Klee and R. Ladner, Qualitative matrices: strong sign-solvability and weak satisfiability, in: H. Greenberg and J. Maybee, eds., Computer-Aided Analysis and Model Simplification (Academic Press, New York, 1981) 293–320.

[4] K. Lancaster, The scope of qualitative economics, Review of Economic Studies 29 (1962) 99–132.

[5] R. Manber, Graph theoretical approach to qualitative solvability of linear systems, Linear Algebra Appl. (1982).

[6] R.B. Marimont, A new method for checking the consistency of precedence matrices, J. Assoc. Comput. Mach. 6 (1959) 164–171.

[7] J. Maybee, Sign solvable graphs, Discrete Appl. Math. 2 (1980) 57–63.

[8] J. Maybee, Sign solvability, in: H. Greenberg and J. Maybee, eds., Computer-Aided Analysis and Model Simplification (Academic Press, New York, 1981) 201–257.

[9] J. Quirk, Qualitative stability of matrices and economic theory: a survey article, in: H. Greenberg and J. Maybee, eds., Computer-Aided Analysis and Model Simplification (Academic Press, New York, 1981) 113–164.

[10] P.A. Samuelson, Foundations of Economic Analysis (Atheneum, New York, 1971; originally published by Harvard University Press, 1947).

[11] R.E. Tarjan, Depth-first search and linear graph algorithms, SIAM J. Comput. 1 (1972) 146–160.